LOW-DISCREPANCY SEQUENCES FOR PIECEWISE SMOOTH FUNCTIONS ON THE TWO-DIMENSIONAL TORUS

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ABSTRACT. We produce explicit low-discrepancy infinite sequences which can be used to approximate the integral of a smooth periodic function restricted to a convex domain with positive curvature in $\mathbb{R}^2$. The proof depends on simultaneous diophantine approximation and a general version of the Erdős-Turán inequality.

Keywords: Koksma-Hlawka inequality, piecewise smooth functions, discrepancy, diophantine approximation, Erdős-Turán inequality.

1. INTRODUCTION

Let f be a suitable function on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and let $\{t(j)\}_{j=1}^N$ be a distribution of points on $\mathbb{T}^d$. The quality of the approximation of $\int_{\mathbb{T}^d} f(t)dt$ by the Riemann sum $N^{-1}\sum_{j=1}^N f(t(j))$ is a basic problem with applications in 2D or 3D computer graphics, and also with applications when d is large (and the curse of dimensionality appears). See e.g. [9]. Any bound of the form

$$\left|\frac{1}{N}\sum_{j=1}^N f(t(j)) - \int_{\mathbb{T}^d} f(t)dt\right| \leq D\left(\{t(j)\}_{j=1}^N\right)V(f)$$

can be termed a *Koksma-Hlawka type inequality*, provided the RHS is a *variation* $V(f)$ of the function f times a *discrepancy* $D\left(\{t(j)\}_{j=1}^N\right)$ of the finite set $\{t(j)\}_{j=1}^N$ with respect to a reasonably simple family of subsets of $\mathbb{T}^d$.

The case $d=1$ is the amazingly simple Koksma inequality, where $\mathbb{T}$ is replaced by the unit interval, $V(f)$ is the usual total variation and $D\left(\{t(j)\}_{j=1}^N\right)$ is the *-discrepancy

$$\sup_{0<\alpha\leq 1}\left|\frac{1}{N}\sum_{j=1}^N \chi_{[0,\alpha)}(t(j)) - \alpha\right| ,$$

that is the discrepancy measured on the family of all intervals anchored at the origin.

See [3], [9], [14], [15], [21], [22], [28] as general references.

The term Koksma-Hlawka inequality properly refers to E. Hlawka's generalization of Koksma inequality to several variables, where f is required to have bounded variation in the sense of Hardy and Krause. In one variable, many familiar bounded functions have bounded variation, but, in several variables, the Hardy-Krause condition cannot be applied to most functions with simple discontinuities. For example: the characteristic function of a polyhedron has bounded Hardy-Krause variation if and only if the polyhedron is a d-dimensional interval.

We recall some of the variants of the Koksma-Hlawka inequality which have appeared in the literature so far. F. Hickernell [20] has proposed Koksma-Hlawka type inequalities

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for reproducing kernel Hilbert spaces. J. Dick [13] has used fractional calculus to prove a Koksma-Hlawka type inequality for functions with relaxed smoothness assumptions. G. Harman [19] has considered a geometric approach and measured the variation by counting the convex sets needed to describe super-level sets of the function f. In [6] the authors of the present paper proposed a Koksma-Hlawka type inequality especially tailored for simplices, while in [7] they have introduced a Koksma-Hlawka type inequality for piecewise smooth functions. Analogues of the above problem in more general settings can be found e.g. in [4] and [5].

K. Basu and A. Owen [1] have recently produced low-discrepancy sequences for a triangle, where the discrepancy is the one considered in [6]. In this paper we propose a sequence of points which gives low discrepancy in the sense of the Koksma-Hlawka type inequality in [7]. We first recall a particular two-dimensional case of the statement therein.

Theorem 1 ([7]). *Let $h(t) = f(t)\chi_\Omega(t)$, where f is a smooth $\mathbb{Z}^2$-periodic function on $\mathbb{R}^2$ and χ_Ω is the characteristic function of a bounded Borel set in $\mathbb{R}^2$. Let*

$$V(f) := 4\|f\|_{L^1(\mathbb{T}^2)} + 2\left\|\frac{\partial f}{\partial t_1}\right\|_{L^1(\mathbb{T}^2)} + 2\left\|\frac{\partial f}{\partial t_2}\right\|_{L^1(\mathbb{T}^2)} + \left\|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_{L^1(\mathbb{T}^2)} .$$

Let $\{t(j)\}_{j=1}^N \subset \mathbb{R}^2$, for any $s \in (0,1)^2$ and for any $x \in \mathbb{R}^2$ let

$$I(s,x) = \cup_{m\in\mathbb{Z}^2}\left([0,s_1]\times[0,s_2]+x+m\right),$$

and let

$$D\left(\{t(j)\}_{j=1}^N\right) := \sup_{s\in(0,1)^2,\, x\in\mathbb{R}^2}\left|\frac{1}{N}\sum_{j=1}^N\sum_{m\in\mathbb{Z}^2}\chi_{I(s,x)\cap\Omega}(t(j)+m) - |I(s,x)\cap\Omega|\right| . \tag{1}$$

Then

$$\left|\frac{1}{N}\sum_{j=1}^N\sum_{m\in\mathbb{Z}^2}h(t(j)+m) - \int_{\mathbb{R}^2}h(t)\,dt\right| \leq V(f)\,D\left(\{t(j)\}_{j=1}^N\right) .$$

Observe that if a set $K \in \mathbb{R}^2$ does not intersect any of its integer translates, then it can be thought of as a subset of $\mathbb{T}^2$, and in that case the expression

$$\frac{1}{N}\sum_{j=1}^N\sum_{m\in\mathbb{Z}^2}\chi_K(t(j)+m) - |K|$$

compares the measure of K with the share of points in K of the collection obtained by projecting $\{t(j)\}_{j=1}^N$ onto $\mathbb{T}^2$. It follows that the above theorem includes, but is slightly more general than the analogous theorem where not only the function f but also the set Ω and the point collection $\{t(j)\}$ are in $\mathbb{T}^2$, and the quantity $D(\{t(j)\}_{j=1}^N)$ is just the discrepancy with respect to the intersection of Ω with all the rectangles in $\mathbb{T}^2$.

We are therefore interested in choices of the set $\{t(j)\}_{j=1}^N$ which give satisfactory upper bounds for the discrepancy (1).

An interesting result in this direction is due to J. Beck [2]: for every positive integer N there is a collection of N points in the unit square with isotropic discrepancy (that is, the discrepancy with respect to all convex sets) bounded by $cN^{-2/3}\log^4 N$. Since the discrepancy (1) is smaller than the isotropic discrepancy, Beck's result gives a sequence that can be used in the Koksma-Hlawka type inequality in Theorem 1 when Ω is convex. On the other hand, Beck's construction is somewhat intricate, and is obtained partly by random and partly by deterministic methods.

A more explicit extensible construction comes from a result of H. Niederreiter (see [23] or [21, page 129 and page 132, Exercise 3.17]): if $1, \alpha, \beta$ are algebraic linearly independent on $\mathbb{Q}$, then the discrepancy of $\{(j\alpha,\, j\beta)\}_{j=1}^{N}$ with respect to all axis parallel rectangles contained in the unit square is bounded by $cN^{-1+\varepsilon}$. This immediately implies that the isotropic discrepancy of this sequence is bounded by $cN^{-1/2+\varepsilon}$ (see [21, Theorem 1.6, page 95]), an estimate that is far from Beck's result.

Our main result is the following.

Theorem 2. *Assume that α, β are real algebraic numbers and that $1, \alpha, \beta$ is a basis of a number field on $\mathbb{Q}$ of degree 3. For all integers $j \geq 0$, let $t(j) = (j\alpha, j\beta)$. Let Ω be a convex domain contained in $\mathbb{R}^2$ with $\mathscr{C}^2$ boundary having strictly positive curvature. Then the discrepancy defined in (1) satisfies*

$$D\left(\{t(j)\}_{j=1}^{N}\right) \leq c\, N^{-2/3} \log N. \tag{2}$$

The above constant c depends on the minimum and the maximum of the curvature of $\partial\Omega$, on its length, and on the numbers α, β.

For example, one can take $\alpha = \xi, \beta = \xi^2$, where ξ is a real root of a third degree irreducible polynomial in $\mathbb{Z}$.

In other words, Theorem 2 says that a regularity assumption on the convex set Ω suffices for the sequence in Niederreiter's result to improve Beck's estimate $N^{-2/3}\log^4 N$. This can be obtained by estimating directly the discrepancy $D(\{t(j)\}_{j=1}^{N})$, and avoiding the isotropic discrepancy. The main tool that will allow us to do it is a version of the Erdős-Turán inequality essentially contained in [11].

2. Proofs and auxiliary results

Let us begin by recalling the above mentioned general form of the Erdős-Turán inequality.

Theorem 3. *There exists a positive function $\psi(u)$ on $[0, +\infty)$ with rapid decay at infinity such that for every collection of points $\{t(j)\}_{j=1}^{N} \subset \mathbb{R}^d$, for every bounded Borel set $D \subseteq \mathbb{R}^d$, and for every $R > 0$,*

$$\begin{aligned} &\left| \frac{1}{N} \sum_{j=1}^{N} \sum_{m \in \mathbb{Z}^2} \chi_D(t(j) + m) - |D| \right| \\ &\leq \left| \widehat{H}_R(0) \right| + \sum_{n \in \mathbb{Z}^2,\, 0<|n|<R} \left(|\widehat{\chi}_D(n)| + \left| \widehat{H}_R(n) \right| \right) \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i n \cdot t(j)} \right|. \end{aligned}$$

Here

$$H_R(x) = \psi(R \operatorname{dist}(x, \partial D)),$$

where dist *is the Euclidean distance in $\mathbb{R}^d$.*

Proof. Take a smooth radial function $m(\xi)$ supported in $|\xi| < 1/2$ and with $\int_{\mathbb{R}^d} m^2(\xi)\, d\xi = 1$, and define

$$\begin{aligned} K(x) &= \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^{-(d+1)/2} (m * m)(\xi)\, e^{2\pi i \xi \cdot x} d\xi, \\ \psi(u) &= e^{2\pi} \left(\int_{|y| \leq 1} K(y)\, dy \right)^{-1} \int_{\{|y| \geq u\}} K(y)\, dy \end{aligned}$$

Since $\widehat{K}(\xi)=0$ if $|\xi|\geq 1$, it follows from the Paley-Wiener theorem that $K(x)$ is an entire function of exponential type smaller than 1, positive with mean 1, all its derivatives have rapid decay at infinity, and $\left|\widehat{K}(\xi)\right|\leq 1$ for every $\xi\in\mathbb{R}^d$. If we set $K_R(x)=R^dK(Rx)$, then the functions

$$
\begin{aligned}
A(x) &= \int_{\mathbb{R}^d} K_R(y)\left(\chi_D(x-y)-H_R(x-y)\right)dy \\
B(x) &= \int_{\mathbb{R}^d} K_R(y)\left(\chi_D(x-y)+H_R(x-y)\right)dy,
\end{aligned}
$$

are entire of exponential type smaller than R and

$$
A(x)\leq\chi_D(x)\leq B(x), \quad |B(x)-A(x)|\leq 4\psi(R\mathrm{dist}(x,\partial D)/2)
$$

(see [11] for the details). Periodization gives

$$
\sum_{m\in\mathbb{Z}^d}A(x+m)\leq\sum_{m\in\mathbb{Z}^d}\chi_D(x+m)\leq\sum_{m\in\mathbb{Z}^d}B(x+m),
$$

and, by the Poisson summation formula,

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^d}A(x+m) &= \sum_{n\in\mathbb{Z}^d}\widehat{K}\left(R^{-1}n\right)\left(\widehat{\chi}_D(n)-\widehat{H}_R(n)\right)e^{2\pi i n\cdot x}, \\
\sum_{m\in\mathbb{Z}^d}B(x+m) &= \sum_{n\in\mathbb{Z}^d}\widehat{K}\left(R^{-1}n\right)\left(\widehat{\chi}_D(n)+\widehat{H}_R(n)\right)e^{2\pi i n\cdot x}
\end{aligned}
$$

are trigonometric polynomials of degree at most R. It now follows that

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\sum_{m\in\mathbb{Z}^d}\chi_D(t(j)+m)-|D| \\
&\leq\frac{1}{N}\sum_{j=1}^{N}\sum_{m\in\mathbb{Z}^d}B(t(j)+m)-|D| \\
&=\frac{1}{N}\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}^d}\widehat{K}\left(R^{-1}n\right)\left(\widehat{\chi}_D(n)+\widehat{H}_R(n)\right)e^{2\pi i n\cdot t(j)}-|D| \\
&=\widehat{H}_R(0)+\sum_{n\in\mathbb{Z}^d,0<|n|<R}\widehat{K}\left(R^{-1}n\right)\left(\widehat{\chi}_D(n)+\widehat{H}_R(n)\right)\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i n\cdot t(j)} \\
&\leq\left|\widehat{H}_R(0)\right|+\sum_{n\in\mathbb{Z}^d,0<|n|<R}\left(|\widehat{\chi}_D(n)|+\left|\widehat{H}_R(n)\right|\right)\left|\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i n\cdot t(j)}\right|.
\end{aligned}
$$

Similar estimates from below can be proved, if one uses $A(x)$ instead of $B(x)$. □

A second tool in the proof is the estimate of the Fourier transform of arcs of curves in $\mathbb{R}^2$. The next two lemmas are well known (see e.g. [27, Chapter 8]). We recall the proof of the first one both to help the unfamiliar reader, and to emphasize its two-dimensional nature.

In what follows, for any arc γ we will denote with $\widehat{\gamma}(\xi)$ the Fourier transform of its arclength measure.

Lemma 4. *Let Ω be a convex set in $\mathbb{R}^2$ with a $\mathscr{C}^2$ boundary with non-vanishing curvature. Let γ be an arc of $\partial\Omega$ and $\kappa_{\min}>0$ be the minimum of the curvature of γ. Then for $|\xi|\geq 1$,*

the Fourier transform is bounded by

$$|\widehat{\gamma}(\xi)| \leq \min\left(\ell, c\frac{1+\kappa_{\min}^{-1/2}}{|\xi|^{1/2}}\right).$$

Here ℓ is the length of the arc and c is a universal constant.

Proof. Let $r(\tau)$ be the parametrization of γ with respect to arclength, so that

$$\widehat{\gamma}(\xi) = \int_0^\ell e^{-2\pi i r(\tau)\cdot\xi} d\tau.$$

For any ξ we have the trivial estimate

$$\left|\int_0^\ell e^{-2\pi i r(\tau)\cdot\xi} d\tau\right| \leq \ell.$$

Assume $\xi \neq 0$ and let

$$\xi = \rho\eta$$

where $|\eta| = 1$ and $\rho > 0$. First consider the (at most) three intervals I_1, I_2 and I_3 where $|r'(\tau)\cdot\eta| > 2^{-1/2}$. By Van der Corput's lemma, since $|r'(\tau)\cdot\eta| > 2^{-1/2}$ and the expression $r''(\tau)\cdot\eta = -\kappa(\tau)\nu(\tau)\cdot\eta$ changes sign at most once (here $\nu(\tau)$ and $\kappa(\tau)$ are respectively the outer normal and the curvature of γ at a point $r(\tau)$), then

$$\left|\int_{I_i} e^{-2\pi i\rho r(\tau)\cdot\eta} d\tau\right| \leq \frac{c_1}{\rho}$$

$(i = 1,2,3)$. The constant c_1 is universal. If $|r'(\tau)\cdot\eta| \leq 2^{-1/2}$ we have $|\nu(\tau)\cdot\eta| \geq 2^{-1/2}$ so that

$$\left|r''(\tau)\cdot\eta\right| = \kappa(\tau)|\nu(\tau)\cdot\eta| > \kappa_{\min}2^{-1/2}.$$

Thus, by Van der Corput's lemma, for the at most three intervals J_1, J_2 and J_3 where $|r'(\tau)\cdot\eta| \leq 2^{-1/2}$, we have

$$\left|\int_{J_j} e^{-2\pi i\rho r(\tau)\cdot\eta} d\tau\right| \leq \frac{c_2}{(\kappa_{\min}\rho)^{1/2}}$$

$(j = 1,2,3)$. Again, c_2 is a universal constant. Finally,

$$\left|\int_0^\ell e^{-2\pi i\rho r(\tau)\cdot\eta} d\tau\right| \leq \min\left(\ell, \frac{3c_1}{\rho} + \frac{3c_2}{(\kappa_{\min}\rho)^{1/2}}\right).$$

When $\rho \geq 1$, this gives

$$\left|\int_0^\ell e^{-2\pi i\rho r(\tau)\cdot\eta} d\tau\right| \leq \min\left(\ell, c\frac{1+\kappa_{\min}^{-1/2}}{\rho^{1/2}}\right).$$

□

Lemma 5. *The Fourier transform of the arclength measure on the segment γ joining two points x and y in $\mathbb{R}^2$ is*

$$\widehat{\gamma}(\xi) = |x-y|\frac{\sin(\pi(x-y)\cdot\xi)}{\pi(x-y)\cdot\xi}e^{-2\pi i\frac{(x+y)}{2}\cdot\xi}.$$

In particular, calling $\ell = |x-y|$ and $\theta = \frac{x-y}{|x-y|}$, we have

$$|\widehat{\gamma}(\xi)| \leq \min\left(\ell, \frac{1}{\pi|\xi\cdot\theta|}\right).$$

Proof. This is just an explicit calculation. □

Before we proceed with the proof of Theorem 2, we need a few results on convex sets in $\mathbb{R}^d$. Let us begin with some terminology.

Definition 6. *Let K be a non-empty compact convex subset (a "convex body") of $\mathbb{R}^d$. The signed distance function δ_K is defined by*

$$\delta_K(x) = \begin{cases} \mathrm{dist}(x, \partial K) & \textit{if } x \in K \\ -\mathrm{dist}(x, \partial K) & \textit{if } x \notin K. \end{cases}$$

For any real number u, define

$$K^u = \left\{ x \in \mathbb{R}^d : \delta_K(x) \geq u \right\}$$

and

$$K_u = \left\{ x \in \mathbb{R}^d : \delta_K(x) = u \right\}$$

The signed distance function is Lipschitz continuous with constant 1, and $|\nabla \delta_K| = 1$ almost everywhere (see [17, Section 14.6]).

Definition 7. *Let B be the closed unit ball centered at the origin. If K is a convex body in $\mathbb{R}^d$, then the outer parallel body of K at distance r is defined as the Minkowski sum of K and rB,*

$$K + rB = \{x + y : x \in K, |y| \leq r\},$$

while the inner parallel body of K at distance r is defined as the Minkowski difference of K and rB,

$$K \div rB = \{x : x + rB \subset K\},$$

Lemma 8. *Let K be a convex body in $\mathbb{R}^d$.*
(i) For any real number u, the set K^u is the outer or the inner parallel body of K at distance $|u|$, according to whether u is negative or positive, that is

$$\begin{aligned} K^u &= K + |u|B, \textit{ if } u \leq 0, \\ K^u &= K \div uB, \textit{ if } u > 0. \end{aligned}$$

(ii) For any real number u, the set K^u is convex (possibly empty).
(iii) If M is a convex body too, then for every $u \geq 0$,

$$(M \cap K)_u = (M_u \cap K^u) \cup (M^u \cap K_u).$$

Proof. Point (i) follows easily from the definitions, while the proof of (ii) can be found in [26, Chapter 3]. As for point (iii), we sketch a proof, highlighting the main steps. First observe that $\partial(K^u) = K_u$ and that $(M \cap K)^u = M^u \cap K^u$ when $u \geq 0$. The thesis now follows after the observation that for any two compact sets A and B one has

$$\partial(A \cap B) = (\partial A \cap B) \cup (A \cap \partial B).$$

□

Lemma 9. *Let K be a convex body in $\mathbb{R}^d$ with $\mathscr{C}^2$ boundary and let $\kappa_{\max}$ be the maximum of all the principal curvatures of ∂K. Finally, let*

$$\Gamma = \Gamma(K, \kappa_{\max}) = \{x : -(2\kappa_{\max})^{-1} < \delta_K(x) < (2\kappa_{\max})^{-1}\}.$$

Then $\delta_K \in \mathscr{C}^2(\Gamma)$. Furthermore, the level set K_u is $\mathscr{C}^2$ whenever $|u| < (2\kappa_{\max})^{-1}$ and its principal curvatures at a point x are given by

$$\kappa_j(x) = \frac{\kappa_j(y)}{1 - u\kappa_j(y)}, \quad j = 1, \dots, d-1,$$

where y is the unique point of ∂K such that $\operatorname{dist}(x,y) = |u|$ and $\kappa_j(y)$ are the principal curvatures of ∂K at y.

Proof. This is essentially a reformulation of Lemmas 14.16 and 14.17 in [17] for the case of convex bodies. □

Let us now move back to the two-dimensional case. In the next two lemmas we estimate the Fourier transforms of the functions χ_D and H_R in Theorem 3, for the specific type of sets D that one needs in the proof of Theorem 2.

Lemma 10. *Let Ω be a convex body in $\mathbb{R}^2$ with $\mathscr{C}^2$ boundary with non-vanishing curvature and let $\kappa_{\min}$ and $\kappa_{\max}$ be the minimum and the maximum of the curvature of $\partial\Omega$. Let I be a rectangle contained in a unit square with sides parallel to the axes, and call $K = \Omega \cap I$. Then there exists a constant c depending only on $\kappa_{\min}$ such that for all $R \geq 4\kappa_{\max}^2$ and for every $n = (n_1, n_2) \in \mathbb{Z}^2$ with $0 < |n| < R$,*

$$|\widehat{H}_R(n)| \leq c\frac{1}{|n|^{3/2}} + c\frac{1}{1+|n_1|}\frac{1}{1+|n_2|}.$$

Here, $H_R(x)$ is the function defined in Theorem 3 by $H_R(x) = \psi(R|\delta_K(x)|)$. Finally, there is a universal constant $c > 0$ such that for all $R \geq 1$,

$$|\widehat{H}_R(0)| \leq \frac{c}{R}.$$

Proof. By the coarea formula (see [16, Theorem 2, page 117]), since $|\nabla\delta_K(x)| = 1$ almost everywhere,

$$\begin{aligned}\widehat{H}_R(n) &= \int_{\mathbb{R}^2} \psi\left(R|\delta_K(x)|\right) e^{-2\pi i x\cdot n} dx \\ &= \int_{-\infty}^{\infty} \psi\left(|Ru|\right) \int_{K_u} e^{-2\pi i x\cdot n} dx du.\end{aligned}$$

where $K_u = \{x : \delta_K(x) = u\}$ as in the above Definition 6, and the integration on the level set K_u is with respect to the Hausdorff measure. Thus

$$\begin{aligned}\left|\widehat{H}_R(n)\right| \leq & \int_{|u|<R^{-1/2}} \psi\left(R|u|\right) du \sup_{|u|<R^{-1/2}} \left|\int_{K_u} e^{-2\pi i x\cdot n} dx\right| \\ & + \int_{|u|\geq R^{-1/2}} \psi\left(R|u|\right) |K_u|\, du \\ \leq & \frac{c_1}{R} \sup_{|u|<R^{-1/2}} \left|\int_{K_u} e^{-2\pi i x\cdot n} dx\right| + \frac{c_2}{R^{10}}.\end{aligned}$$

The constant c_1 is just the integral of 2ψ on $[0, +\infty)$, while c_2 depends on the rapid decay of ψ and the slow growth of $|K_u|$ (recall that K^u is convex and contained in a square of side $1 + 2|u|$, and therefore the Hausdorff measure of K_u is smaller than $4(1+2|u|)$). In particular, c_1 and c_2 are universal constants and we immediately have that for any $R \geq 1$

$$\left|\widehat{H}_R(0)\right| \leq \frac{c}{R},$$

where c is a universal constant.

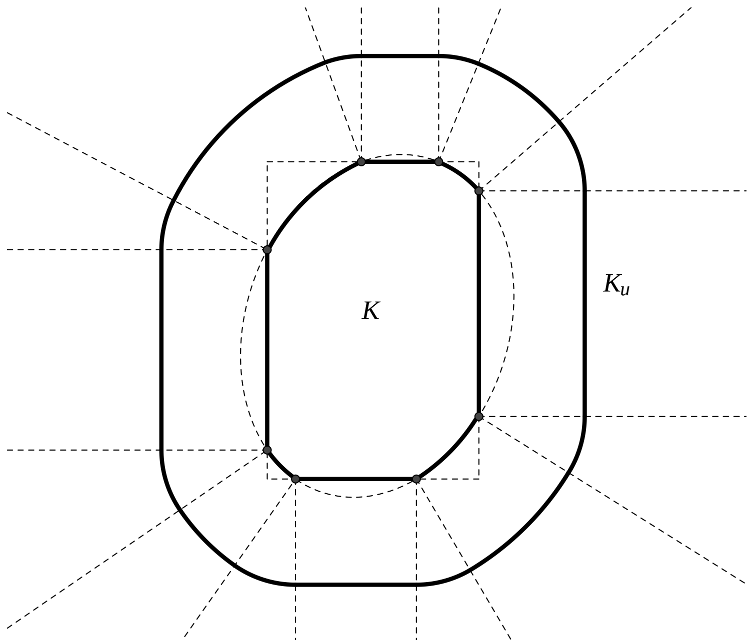

FIGURE 1. A convex body $K = \Omega \cap I$ and the relative set K_u, with $u < 0$. Ω has smooth boundary with non vanishing curvature and I is a rectangle.

Now assume $n \neq 0$, $R^{-1/2} \leq 1/(2\kappa_{\max})$ and $0 \leq u \leq R^{-1/2}$. Then, by the above Lemma 8 and Lemma 9, K_u consists of at most four smooth convex curves with curvature bounded below by $\kappa_{\min}$, and at most four segments of length at most 1 parallel to the axes. By Lemma 4 and Lemma 5 this gives

$$\sup_{0 \leq u \leq R^{-1/2}} \left| \int_{K_u} e^{-2\pi i x \cdot n} dx \right| \leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1+|n_i|},$$

where the constant c depends only on the curvature $\kappa_{\min}$. On the other hand, if $R^{-1/2} \leq 1/(2\kappa_{\max})$ and if $-R^{-1/2} \leq u < 0$, then K_u is composed by at most four smooth convex curves with curvature greater than or equal to $2\kappa_{\min}/3$, at most four segments parallel to the axes and of length at most 1, and at most eight arcs of circles of radius $|u|$. In order to better understand this, observe (see Figure 1) that one can divide the complement of K into at most sixteen regions by taking the two normals to ∂K at each "vertex" of K (there are at most eight "vertices"). The part of K_u that intersects a region attached to a straight line is a parallel straight line of length at most 1. The part of K_u that intersects a region attached to a curve coming from $\partial\Omega$ is a part of Ω_u. Finally, the part of K_u that intersects a region attached to a vertex of K is an arc of circle of radius $|u|$.

It follows that

$$\begin{aligned} \sup_{-R^{-1/2} \leq u < 0} \left| \int_{K_u} e^{-2\pi i x \cdot n} dx \right| &\leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1+|n_i|} + c \frac{|u|^{1/2}}{|n|^{1/2}} \\ &\leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1+|n_i|}, \end{aligned}$$

where the constant c depends only on the minimal curvature $\kappa_{\min}$. Therefore, when $0 < |n| < R$ we have

$$\begin{aligned}\left|\widehat{H}_R(n)\right| &\leq c\frac{1}{R}\frac{1}{|n|^{1/2}} + c\sum_{i=1}^{2}\frac{1}{R}\frac{1}{1+|n_i|} + c\frac{1}{R^{10}} \\ &\leq c\frac{1}{|n|^{3/2}} + c\frac{1}{1+|n_1|}\frac{1}{1+|n_2|}.\end{aligned}$$

□

Lemma 11. *Let Ω be a convex body in $\mathbb{R}^2$ with $\mathscr{C}^2$ boundary with non-vanishing curvature and let $\kappa_{\min}$ be the minimum of the curvature of $\partial\Omega$. Let I be a rectangle contained in a unit square with sides parallel to the axes, and call $K = \Omega \cap I$. Then there exists a constant c depending only on $\kappa_{\min}$ such that for every $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$*

$$|\widehat{\chi}_K(n)| \leq c\frac{1}{|n|^{3/2}} + c\frac{1}{1+|n_1|}\frac{1}{1+|n_2|}.$$

Proof. An application of the divergence theorem gives

$$\begin{aligned}|\widehat{\chi}_K(n)| &= \left|\int_K e^{-2\pi i n\cdot x}dx\right| = \left|\int_{\partial K}\frac{\nu(x)\cdot n}{2\pi i|n|^2}e^{-2\pi i n\cdot x}dx\right| \\ &= \frac{1}{2\pi|n|}\left|\int_{\partial K}\nu(x)\cdot\frac{n}{|n|}e^{-2\pi i n\cdot x}dx\right|.\end{aligned}$$

Here $\nu(x)$ is the outer normal to ∂K at the point x. This oscillatory integral can be estimated by means of standard techniques. We include the details for the sake of completeness. The boundary of $K = \Omega \cap I$ is composed by at most four smooth convex curves with curvature bounded below by $\kappa_{\min}$, coming from $\partial\Omega$, and at most four segments of length at most 1 parallel to the axes, coming from ∂I. We therefore split the above integral into a sum of integrals over the components of ∂K described above. When integrating over a segment, the quantity $\nu(x)\cdot n/|n|$ remains constant and an immediate application of Lemma 5 gives the estimate

$$c\frac{1}{1+|n_1|}\frac{1}{1+|n_2|},$$

with c a universal constant. Let us now estimate the integral over an arc of $\partial\Omega$, call it γ. If $r(\tau)$ is a parametrization of γ with respect to arclength, integration by parts gives

$$\begin{aligned}&\left|\int_\gamma \nu(x)\cdot\frac{n}{|n|}e^{-2\pi i n\cdot x}dx\right| = \left|\int_0^\ell \nu(r(\tau))\cdot\frac{n}{|n|}e^{-2\pi i n\cdot r(\tau)}d\tau\right| \\ =\ & \left|\nu(r(\ell))\cdot\frac{n}{|n|}\int_0^\ell e^{-2\pi i n\cdot r(u)}du - \int_0^\ell \frac{d}{d\tau}(\nu(r(\tau)))\cdot\frac{n}{|n|}\int_0^\tau e^{-2\pi i n\cdot r(u)}du\,d\tau\right| \\ \leq\ & \left|\int_0^\ell e^{-2\pi i n\cdot r(u)}du\right| + \int_0^\ell \kappa(\tau)d\tau \sup_{0\leq\tau\leq\ell}\left|\int_0^\tau e^{-2\pi i n\cdot r(u)}du\right| \\ \leq\ & \left|\int_0^\ell e^{-2\pi i n\cdot r(u)}du\right| + 2\pi\sup_{0\leq\tau\leq\ell}\left|\int_0^\tau e^{-2\pi i n\cdot r(u)}du\right| \leq \frac{c}{|n|^{1/2}}.\end{aligned}$$

Here $\kappa(\tau)$ is the curvature of γ at the point $r(\tau)$ and $\int_0^\ell \kappa(\tau)d\tau$ is the total curvature of γ. Since γ is an arc of $\partial\Omega$, the total curvature of γ is smaller than the total curvature of $\partial\Omega$, that is 2π. The last inequality is just an immediate application of Lemma 4, where the constant c above depends only on the minimal curvature $\kappa_{\min}$ of $\partial\Omega$.

□

We are now ready to proceed with the proof of the main result of the paper.

Proof of Theorem 2. Let $\kappa_{\min}$ and $\kappa_{\max}$ be the minimum and the maximum of the curvature of $\partial\Omega$. If we call $m_1, \ldots, m_q$ the lattice points for which the sets

$$([0,s_1] \times [0,s_2] + x + m_i) \cap \Omega$$

are nonempty, and let

$$K_i = ([0,s_1] \times [0,s_2] + x + m_i) \cap \Omega,$$

then of course

$$\cup_{m\in\mathbb{Z}^2} (([0,s_1] \times [0,s_2] + x + m) \cap \Omega) = \cup_{i=1}^{q} K_i.$$

The number q is bounded by the maximum number of unit squares with integer vertices that intersect any given translate of Ω in $\mathbb{R}^2$. This number is of course bounded by $(\operatorname{diam}(\Omega) + 2)^2$. We recall that we need a uniform estimate with respect to s and x.

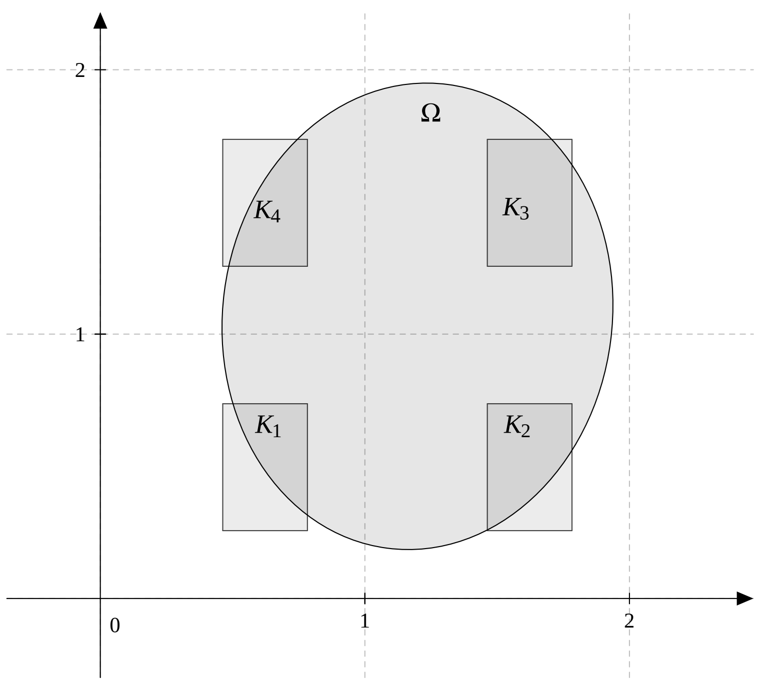

FIGURE 2. The intersection of a convex set Ω with smooth boundary having non-vanishing curvature with the integer translates of a fixed rectangle.

The sets K_i are as in Figure 2; at most four sides are parallel to the coordinate axes, while the curved parts come from $\partial\Omega$. The discrepancy

$$\left| \frac{1}{N} \sum_{j=1}^{N} \sum_{m\in\mathbb{Z}^2} \chi_{I(s,x)\cap\Omega} (t(j) + m) - |I(s,x) \cap \Omega| \right|$$

is clearly bounded by the sum of the discrepancies of the sets K_i,

$$\sum_{i=1}^{q}\left|\frac{1}{N}\sum_{j=1}^{N}\sum_{m\in\mathbb{Z}^2}\chi_{K_i}(t(j)+m)-|K_i|\right|,$$

and we shall therefore study the discrepancy of a single piece K_i. Let us call K one such set.

By the general form of the Erdős-Turán inequality in Theorem 3, the discrepancy of a single piece K is bounded by the quantity

$$\left|\widehat{H}_R(0)\right|+\sum_{n\in\mathbb{Z}^2,\,0<|n|<R}\left(|\widehat{\chi}_K(n)|+\left|\widehat{H}_R(n)\right|\right)\left|\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i n\cdot t(j)}\right|. \tag{3}$$

We recall that $R>0$ is a number that we can choose at our convenience, $H_R(x)=\psi(R|\delta_K(x)|)$ and $\psi(u)$ is a properly chosen function on $[0,+\infty)$ with rapid decay at infinity.

The estimates of $\widehat{\chi}_K(n)$ and $\widehat{H}_R(n)$ are contained in the above Lemmas 10 and 11, while the estimate of the exponential sums follows a standard argument,

$$\left|\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i n\cdot(j\alpha,j\beta)}\right|=\left|\frac{1}{N}\sum_{j=1}^{N}e^{2\pi i jn\cdot(\alpha,\beta)}\right|$$
$$=\left|\frac{1}{N}\frac{\sin(\pi Nn\cdot(\alpha,\beta))}{\sin(\pi n\cdot(\alpha,\beta))}\right|\leq\frac{1}{N\|n\cdot(\alpha,\beta)\|},$$

where $\|u\|$ is the distance from u to the closest integer.

Overall, the goal estimate (3) becomes

$$\frac{1}{R}+\sum_{0<|n|<R}\left(\frac{1}{|n|^{3/2}}+\frac{1}{1+|n_1|}\frac{1}{1+|n_2|}\right)\frac{1}{N\|n\cdot(\alpha,\beta)\|}.$$

Observe now that

$$\sum_{0<|n|<R}\frac{1}{1+|n_1|}\frac{1}{1+|n_2|}\frac{1}{\|n\cdot(\alpha,\beta)\|}$$
$$\leq c\sum_{i=0}^{\log R}\sum_{j=0}^{\log R}\frac{1}{2^i}\frac{1}{2^j}\sum_{n_1=2^i}^{2^{i+1}-1}\sum_{n_2=2^j}^{2^{j+1}-1}\frac{1}{\|n_1\alpha+n_2\beta\|}$$
$$+c\sum_{i=0}^{\log R}\frac{1}{2^i}\sum_{n_1=2^i}^{2^{i+1}-1}\frac{1}{\|n_1\alpha\|}+c\sum_{j=0}^{\log R}\frac{1}{2^j}\sum_{n_2=2^j}^{2^{j+1}-1}\frac{1}{\|n_2\beta\|}.$$

Let us study the sum $\sum_{n_1=2^i}^{2^{i+1}-1}\sum_{n_2=2^j}^{2^{j+1}-1}\|n_1\alpha+n_2\beta\|^{-1}$ first. By the celebrated result of W. M. Schmidt [24], see also [25, Theorem 7C], since $1,\alpha,\beta$ are linearly independent on $\mathbb{Q}$, for any $\varepsilon>0$ there is a constant $\gamma>0$ such that for any $n\neq 0$,

$$\|n_1\alpha+n_2\beta\|>\frac{\gamma}{(1+|n_1|)^{1+\varepsilon}(1+|n_2|)^{1+\varepsilon}}. \tag{4}$$

Then, following [12], in any interval of the form

$$\left[\frac{(k-1)\gamma}{(1+2^{i+1})^{1+\varepsilon}(1+2^{j+1})^{1+\varepsilon}},\frac{k\gamma}{(1+2^{i+1})^{1+\varepsilon}(1+2^{j+1})^{1+\varepsilon}}\right),$$

where k is a positive integer, there are at most two numbers of the form $\|n_1\alpha+n_2\beta\|$, with $2^i\leq n_1<2^{i+1}$ and $2^j\leq n_2<2^{j+1}$. Indeed, assume by contradiction that there are three such numbers. Then for two of them, say $\|n_1\alpha+n_2\beta\|$ and $\|m_1\alpha+m_2\beta\|$, the fractional

parts of $n_1\alpha+n_2\beta$ and $m_1\alpha+m_2\beta$ belong either to $(0,1/2]$ or to $(1/2,1)$. Assume without loss of generality that they belong to $(0,1/2]$. Then

$$\begin{aligned}\frac{\gamma}{(1+2^{i+1})^{1+\varepsilon}(1+2^{j+1})^{1+\varepsilon}} &> |\|n_1\alpha+n_2\beta\|-\|m_1\alpha+m_2\beta\||\\ &= |n_1\alpha+n_2\beta-p-(m_1\alpha+m_2\beta-q)|\\ &\geq \|(n_1-m_1)\,\alpha+(n_2-m_2)\,\beta\|\\ &> \frac{\gamma}{(1+2^{i+1})^{1+\varepsilon}(1+2^{j+1})^{1+\varepsilon}}.\end{aligned}$$

By the same type of argument, in the first interval $\left[0,\frac{\gamma}{(1+2^{i+1})^{1+\varepsilon}(1+2^{j+1})^{1+\varepsilon}}\right)$, there are no points of the form $\|n_1\alpha+n_2\beta\|$. It follows that

$$\sum_{n_1=2^i}^{2^{i+1}-1}\sum_{n_2=2^j}^{2^{j+1}-1}\frac{1}{\|n_1\alpha+n_2\beta\|}\leq c\sum_{k=1}^{2^{i+j}}\frac{2^{(i+j)(1+\varepsilon)}}{k\gamma}\leq c2^{(i+j)(1+\varepsilon)}\,(i+j)\,.$$

Similarly,

$$\sum_{n_1=2^i}^{2^{i+1}-1}\frac{1}{\|n_1\alpha\|}\leq c2^{i(1+\varepsilon)}i,\qquad \sum_{n_2=2^j}^{2^{j+1}-1}\frac{1}{\|n_2\beta\|}\leq c2^{j(1+\varepsilon)}j.$$

Finally,

$$\begin{aligned}&\sum_{0<|n|<R}\frac{1}{(1+|n_1|)}\frac{1}{(1+|n_2|)}\frac{1}{\|n\cdot(\alpha,\beta)\|}\\ &\leq c\sum_{i=0}^{\log R}\sum_{j=0}^{\log R}\frac{1}{2^i}\frac{1}{2^j}2^{(i+j)(1+\varepsilon)}\,(i+j)+c\sum_{i=0}^{\log R}\frac{1}{2^i}2^{i(1+\varepsilon)}i+c\sum_{j=0}^{\log R}\frac{1}{2^j}2^{j(1+\varepsilon)}j\\ &\leq c\sum_{i=0}^{\log R}2^{i\varepsilon}R^{\varepsilon}\log R+cR^{\varepsilon}\log R\leq cR^{2\varepsilon}\log R.\end{aligned}$$

Finally, we use the hypothesis that $1,\alpha,\beta$ are a basis of a number field in $\mathbb{Q}$. By a simple argument in number field theory, there is a constant η such that for any $n\neq 0$,

$$\|n_1\alpha+n_2\beta\|>\frac{\eta}{(\max(|n_1|,|n_2|))^2}.$$

See for example [25, Theorem 6F]. By a similar argument as before, this implies that

$$\sum_{\max(|n_1|,|n_2|)=2^i}^{2^{i+1}-1}\frac{1}{\|n\cdot(\alpha,\beta)\|}\leq c\sum_{k=1}^{2^{2i}}\frac{2^{2i}}{k}\leq ci2^{2i}.$$

Thus,

$$\begin{aligned}\sum_{0<|n|<R}\frac{1}{|n|^{3/2}}\frac{1}{\|n\cdot(\alpha,\beta)\|}&\leq c\sum_{i=0}^{\log R}\sum_{\max(|n_1|,|n_2|)=2^i}^{2^{i+1}-1}\frac{1}{|n|^{3/2}}\frac{1}{\|n\cdot(\alpha,\beta)\|}\\ &\leq c\sum_{i=0}^{\log R}\frac{1}{2^{3i/2}}i2^{2i}\leq cR^{1/2}\log R.\end{aligned}$$

Setting $R=N^{2/3}$ gives the desired estimate $N^{-2/3}\log N$, as long as $N\geq 8\kappa_{\max}^3$. $\square$

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